Capillary Viscometer:

Consider cylindrical coordinates, laminar flow driven by pressure so that a differential fluid element of thickness dr and length L is considered. The flow is pressure driven by a pressure drop of ΔP. In cylindrical coordinates the only non-zero components of the Navier-Stokes equation are,

$$\frac{-\partial P}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\tau_{rz}(r)\right)$$

This equation is integrated by parts with the limits given above at the ends of the capillary and knownign that there is a finite shear stress (albeit 0) at the center of the capillary where r = 0. Then,

$$\tau_{rz}(r) = \frac{\Delta P}{L}\frac{r}{2} = \tau_R\left(\frac{r}{R}\right)$$

so the shear stress is linear in r and is equal to 0 at the center of the capillary.

The simplest way to obtain the shear rate is to take the derivative of the velocity. For a Newtonian fluid Poiseiulle's Law for capillary flow is,

$$v_z(r) = \frac{2Q}{\pi R^2}\left[1-\left(\frac{r}{R}\right)^2\right]$$

The derivative yields,

$$\dot{\gamma}_{rz} = \frac{4Q}{\pi R^3}\left(\frac{r}{R}\right) = \dot{\gamma}_R\left(\frac{r}{R}\right) = \dot{\gamma}_{apparent}\left(\frac{r}{R}\right)$$

where the subscript "apparent" is used below for non-Newtonian behavior.

The ratio of the shear stress at the wall (r = R) and the rate of strain at the wall yields the Newtonian viscosity,

$$\eta = \frac{\Delta PR}{2L}\left(\frac{\pi R^3}{4Q}\right)$$

This can be obtained from the slope of the r = R values of rate of strain versus shear stress. This plot will also indicate problems with the Newtonian assumption (curve isn't linear) and with end effects, i.e. that L/R >>> 1 (intercept isn't at 0,0).

If the fluid is non-Newtonian and the functional form for viscosity is known a modified Poiseuille equation can be used. For example, a power-law fluid follows a velocity profile given by,

$$v_z(r) = R^{1/n+1}\left(\frac{\Delta P}{2mL}\right)^{\frac{1}{n}}\left(\frac{1}{1+1/n}\right)\left(1-\left(\frac{r}{R}\right)^{1+1/n}\right)$$

The constitutive equation,

$$\tau_{rz} = m\dot{\gamma}_{rz}^{n}$$

combined with the derivative of the PLF Poiseuille equation yields,

$$\dot{\gamma}_R = \left[\frac{-\partial v_z}{\partial r}\right]_R = \left(\frac{\tau_R}{m}\right)^{1/n} = \dot{\gamma}_{apparent}\left(\frac{3+1/n}{4}\right)$$

a log-log plot of the r = R values for the apparent rate of strain (calculated Newtonian) versus the shear rate yields a slope of 1/n and an intercept of $\frac{4m^{-1/n}}{3+(1/n)}$.

The velocity profile for a power-law fluid is much flatter than the parabolic shape for a Newtonian fluid and is often approximated by "plug-flow" which is a flat velocity profile.

For the general case where a function for v(r) is not known it is necessary to calculate the rate of strain at r = R from the volumetric flow rate, Q. The goal is to determine a relationship between the rate of strain at r = R and the shear stress at the same position. This was derived by Rabinowitsch and the function that results is the Rabinowitsch equation. Q is defined by,

$$Q = 2\pi\int_{r=0}^{r=R} r v_z(r)\,dr$$

Integration by parts and substitution of the rate of strain for $-dv_z/dr$ yields,

$$Q = \pi\int_{0}^{R}\dot{\gamma}_{rz}r^2\,dr$$

under the assumption that v(r) is 0 at r = R and that the rate of strain is 0 at r = 0. Next a substitution is made for r and dr using the original expression for the rate of strain,

$$r = \frac{\tau_{rz}(r)}{\tau_R}R$$

then,

$$Q = \frac{\pi R^3}{\tau_R^3} \int_{\tau=0}^{\tau=\tau_R} \dot{\gamma}(\tau)\tau^2 d\tau$$

knowing the relationship between the apparent rate of strain and Q, and using this expression for Q we have,

$$\dot{\gamma}_{apparent} = \frac{4Q}{\pi R^3} = \frac{4}{\tau_R^3} \int_{\tau=0}^{\tau=\tau_R} \dot{\gamma}(\tau)\tau^2 d\tau$$

or

$$\tau_R^3 \dot{\gamma}_{apparent} = 4 \int_{\tau=0}^{\tau=\tau_R} \dot{\gamma}(\tau)\tau^2 d\tau$$

using the Lebinitz Rule,

$$\frac{d}{d\tau_R}\left(\dot{\gamma}_{apparent}\tau_R^3\right) = 4 \int_{\tau=0}^{\tau=\tau_R} \frac{\partial}{\partial \tau_R}\left[\dot{\gamma}(\tau)\tau^2\right] d\tau + 4\dot{\gamma}(\tau_R)\tau_R^2$$

The first term on the right is 0 since the rate of strain is 0 at r = 0 and the velocity is 0 at r = R. Using d(lnx) = dx/x, this function can be reduced to,

$$\dot{\gamma}(\tau_R) = \frac{\dot{\gamma}_{apparent}}{4}\left[3 + \frac{d\left(\ln \dot{\gamma}_{apparent}\right)}{d\left(\ln \tau_R\right)}\right]$$

This is one form of the Rabinowitsch Equation. A plot of the log of the apparent rate of strain versus the log of the rate of strain at R yields a local slope at r that can be used to modify the apparent rate of strain for a Newtonian fluid to yield the actual rate of strain.

The Rabinowitsch equation can be used to predict the rate of strain for a power law fluid, for instance.

$$\tau_{rz} = m\dot{\gamma}_{rz}^n \textbf{ and } \ln\tau_{rz} = \ln m + n \ln\dot{\gamma}_{rz}$$

so

$$\frac{d \ln \dot{\gamma}_{rz}}{d \ln \tau_{rz}} = \frac{1}{n}$$

and,

$$\dot{\gamma}(\tau_R) = \frac{\dot{\gamma}_{apparent}}{4}\left[3 + \frac{d\left(\ln \dot{\gamma}_{apparent}\right)}{d\left(\ln \tau_R\right)}\right] = \frac{\dot{\gamma}_{apparent}}{4}\left[3 + (1/n)\right] =$$

which is the same as the equation derived above from the velocity profile. The viscosity using the Rabinowitsch equation is given by,

$$\eta = \frac{4\tau_R}{\dot{\gamma}_{apparent}} \left[3 + \frac{d\left(\ln \dot{\gamma}_{apparent}\right)}{d\left(\ln \tau_R\right)} \right]^{-1}$$

Corrections to the Pouiseuille equation for measurement of melt index involve consideration of end effects (Bagley Correction) and slip at the capillary wall (Mooney Analysis) as discussed in F. A. Morrison's text "Understanding Rheology" pp. 393 to 397.

The capillary viscometer has a number of limitations which are listed below:

1) The rate of strain can not be easily manipulated in a continuous manner.
2) It is not possible to measure the normal stress differences.
3) Dynamic measurements are not possible.
4) If only a single measurement is made you need a model for the velocity profile to calculate the viscosity.
5) For the most part you need a constitutive equation to measure the constitutive equation.

These drawbacks can be mitigated in other viscometers, however, the capillary viscometer is extremely easy to use, is not subject to major experimental error and is the least expensive instrument for the measurement of viscosity. For these reasons it is commonly used in industry to obtain the melt index to roughly characterize the flow properties of polymers. Additionally, the flow geometry in a capillary viscometer mimics flow in the die of the extruder as well as in the runners of injection molding and in several other processing flows.