

## Yield Criteria for Ductile Materials and Fracture Mechanics of Brittle Materials

Brittle materials are materials that display Hookean behavior (linear relationship between stress and strain) and which fail at a discrete strain. Plastic materials display a variety of yielding behaviors associated with changes in the internal structure of the material at a fixed level of strain and at a fixed rate of strain. The time dependence to yielding behavior is due to the structural changes that must occur in order for the material to yield. Then an understanding of yielding in plastic materials requires a detailed understanding of the microstructure of a material and the mechanisms available to the microstructure to deform and absorb energy. A ductile material displays a simple plastic yielding behavior characterized by a gradual and non-discrete reduction in the slope of the stress strain curve. Ductile yielding can not be recovered by release of the stress, that is, it is a permanent deformation. Ductile behavior is observed for simple metal microstructures such as aluminum and is used for cold drawing of metals. Several simple continuum models (ignoring microstructure) have been proposed to predict (not understand) yielding behavior in simple ductile materials.

### Von Mises' Criterion:

The stress applied to a material can be broken into the hydrostatic pressure and the deviatoric stress,  $\sigma'_{ij}$ ,

$$\sigma'_{ij} = \begin{matrix} \frac{2}{3} \sigma_x - \frac{1}{3} \sigma_y - \frac{1}{3} \sigma_z & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \frac{2}{3} \sigma_y - \frac{1}{3} \sigma_x - \frac{1}{3} \sigma_z & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \frac{2}{3} \sigma_z - \frac{1}{3} \sigma_y - \frac{1}{3} \sigma_x \end{matrix} = \sigma_{ij} - p \delta_{ij}$$

Since yield is a bulk property not associated with the coordinate frame we can consider that a yield criterion could be based on the invariants of the deviatoric stress. The deviatoric stress has 3 invariants,  $J_1$ ,  $J_2$  and  $J_3$ . The first is related to the hydrostatic stress which is probably not associated with yield. A simple model might involve the second invariant only. This is the approach of the Von Mises' yield criterion.

$$J_2 = \frac{1}{6} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] = k^2,$$

written in terms of the principle stresses and where  $k$  is a constant. When  $J_2$  exceeds  $k^2$  yielding occurs. In uniaxial tension  $\sigma_1 = s$  and  $\sigma_2 = \sigma_3 = 0$  so,

$$s_0 = (\sqrt{3})k$$

### Tresca Criterion:

Another approach is to consider that yielding is associated with shear stress. The maximum shear stress can be calculated from the principle stresses where 1 is the largest and 3 the smallest principle stress,

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_0}{2} = k$$

where  $\sigma_0$  is for uniaxial tension and k is a constant above which yield occurs.

This can be written in terms of both the second and third invariant of the deviatoric stress in a messy expression.

### Brittle Fracture of Materials:

A material fractures when two new surfaces are created under tension, for instance. On an atomic scale fracture requires the separation of atoms or molecules in the bulk solid. We can begin a consideration of fracture by considering the energy, cohesive energy, that holds a solid together at the atomic level. Atoms are held at a fixed distance because there is a repulsive component of cohesive energy associated with atoms approaching each other and an attractive component of cohesive energy associated with atoms being separated. The force on an atom is a combination of these repulsive and attractive forces that offset each other at the average atomic separation distance,  $a_0$ . In order for two new surfaces to form the attractive part of the cohesive energy must be overcome by a tensile load. If we assume that the cohesive force on an atom can be represented by a sine curve then,

$$F = F_{\max} \sin \frac{2\pi x}{a_0} \quad F_{\max} \frac{2\pi x}{a_0} = \frac{Ex}{a_0}$$

where  $a_0$  is the repeat distance for atoms, x is the displacement, E is the elastic modulus and  $F_{\max}$  is the maximum in the cohesive force that holds an atom at a fixed location by a balance of attractive and repulsive forces.  $F_{\max}$  is associated with the maximum possible strength for a material,

$$F_{\max} = \frac{E}{2} \frac{E}{a_0}$$

where the latter equality assumes  $a_0 = 2 a_0$ .

The net energy change,  $U_0$ , in brittle failure is associated with the formation of two new surfaces with a surface energy  $\gamma_s$ . This can be equated with the area under the stress displacement curve,

$$U_0 = \int_0^{l/2} \sigma_{\max} \sin \frac{2\pi x}{\lambda} dx = \frac{\sigma_{\max}^2}{2} = 2 \sigma_{\max}^2$$

$$= \frac{2 \sigma_{\max}^2}{\lambda}$$

so  $\sigma_{\max} = \frac{E \epsilon_s^{1/2}}{a_0}$

This yields a fracture strength that is generally 10 to 100 times the observed fracture strength for most materials. Defect free materials approach this theoretical strength, i.e. small fibers or nanocrystals. For the vast majority of materials brittle failure is associated with innate flaws that produce small cracks. The stress concentration associated with such a flaw leads to brittle failure in all commonly encountered brittle materials.

Consider a crack of length  $2c$  with a rounded tip with a tip radius of curvature of  $r$ . For an elliptical crack the stress concentration leads to a maximum stress at the tip of the elliptical hole in a flat sheet of infinite width,

$$\sigma_{\max} = \sigma \left( 1 + 2 \frac{a}{b} \right)$$

which reduces to  $3\sigma$  for a circular hole. For a crack a similar function can be derived,

$$\sigma_{\max} = \sigma \left( 1 + 2 \frac{c}{r} \right) = 2 \sigma \frac{c}{r}$$

The fracture stress for the bulk material can be calculated based on the native flaw size,  $2c$ , and the curvature of the native flaw tip,  $r$ , and assuming that crack propagation occurs when the concentrated stress at the crack tip exceeds the material's theoretical fracture strength (above),

$$\sigma_{\max} = \frac{E \epsilon_s^{1/2}}{4a_0 c} = \frac{E \epsilon_s^{1/2}}{4c}$$

where the latter equality assumed that the smallest crack tip radius is on the order of the atomic spacing,  $a_0$ .

### Griffith Theory for Brittle Fracture:

When a material is strained the integral under the stress strain curve yields the elastic strain energy. Failure of the material could be thought of as resulting from a conversion of this elastic strain energy to the formation of two new surfaces with surface energy  $\gamma_s$ . Griffith theory assumes that native flaws exist in a material and calculates the stress necessary for the growth of

these flaws to cracks and to failure based on this energy balance between elastic strain energy and surface energy.

For a crack of length  $2c$  in a thin plate (plane stress) the change in strain energy associated with the formation of a crack is given by,

$$U_E = -\frac{c^2 \sigma^2}{E}$$

The surface energy created by the crack is given by,

$$U_s = 4c \gamma_s$$

Then the change in energy associated with the presence of a crack is given by,

$$U = U_s + U_E = 4c \gamma_s - \frac{c^2 \sigma^2}{E}$$

we set the derivative of this change in energy with respect to the crack length,  $c$ , to 0 in order to find the equilibrium state for a crack of length  $c$ ,

$$\frac{dU}{dc} = 4\gamma_s - \frac{2c\sigma^2}{E} = 0$$

and

$$c^* = \frac{2E\gamma_s}{\sigma^2}^{1/2}$$

For a 3-d sample under tensile stress the Griffith equation can be modified,

$$c^* = \frac{2E\gamma_s}{(1-\nu^2)\sigma^2}^{1/2}$$

The scaling form of the Griffith equation is, for the most part, considered generally applicable to brittle Hookean materials,

$$c^* \propto \frac{E\gamma_s}{\sigma^2}^{1/2}$$

So we can say, for instance, that increasing the modulus by a factor of 4 increases the fracture strength by a factor of 2 or reduction in the surface energy by a factor of 4 reduces the fracture strength by a factor of 2. The latter is commonly used to enhance fracture in glasses, e.g. to break a glass rod you would score the rod and apply water to reduce the surface energy at the score tip.

**Plastic Deformation in Brittle Failure:**

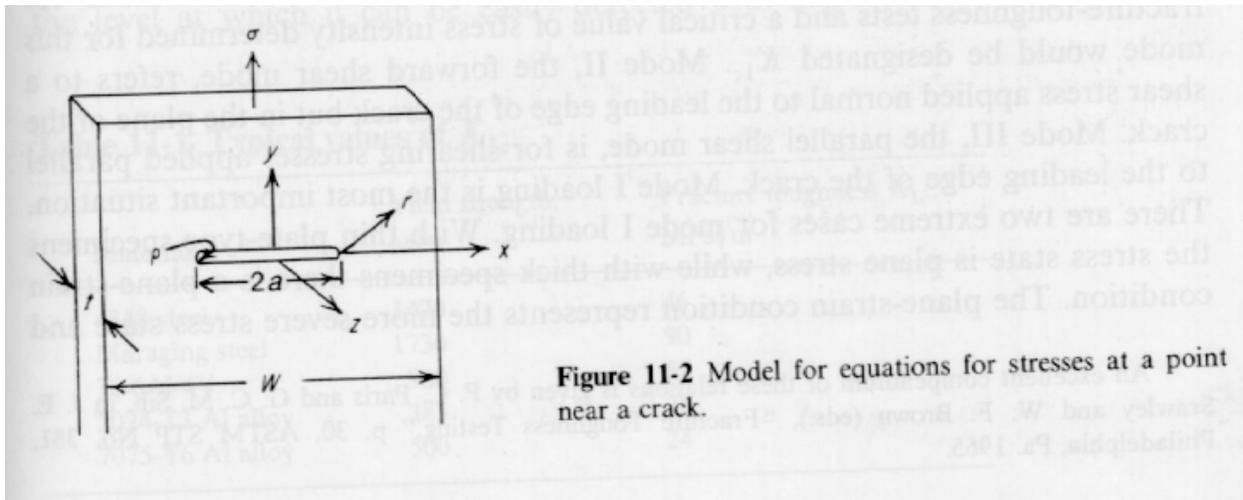
Metals and plastics generally display significant plastic (non-reversible) deformation at the crack tip that is not accounted for in the Griffith equation. The simplest approach to modification of the Griffith equation for plastic deformation is to include a constant,  $\rho$ , that linearly adds to the surface energy term and accounts for a constant amount of plastic deformation per unit of created surface area in crack growth,

$$* = \frac{2E(s + \rho)}{c} \quad \frac{E_p}{c} \quad = \quad \frac{EG_c}{a} \quad ^{1/2}$$

where intermediate equality indicates that the plastic deformation energy is generally much larger than the surface energy. The latter expression is due to Irwin and  $G_c$  is the "critical value of the crack extension force" and  $a$  is a more common term for half the crack length referred to as  $c$  in the previous discussion,

$$G = \frac{a^2}{E}$$

$G$  is also called "the strain-energy release rate" since it reflects the rate of transfer of energy from elastic deformation to irreversible deformation of the material at the crack tip as the crack grows.  $G_c$  is also called the fracture toughness, i.e. the strain-energy release rate at failure where failure is defined as growth of a crack.  $G$  can be determined by measurements of crack propagation under stress.



From Dieter.

The stress distribution at the crack tip shown in Dieter's figure 11-2 is given by,

$$\begin{aligned}
 x &= \frac{a}{2r}^{1/2} \cos\frac{\theta}{2} \left( 1 - \sin\frac{\theta}{2} \sin\frac{3\theta}{2} \right) \\
 y &= \frac{a}{2r}^{1/2} \cos\frac{\theta}{2} \left( 1 + \sin\frac{\theta}{2} \sin\frac{3\theta}{2} \right) \\
 xy &= \frac{a}{2r}^{1/2} \sin\frac{\theta}{2} \cos\frac{\theta}{2} \cos\frac{3\theta}{2}
 \end{aligned}$$

where  $\theta$  is the angle in the plane of the sample off the crack axis. For  $\theta = 0$ ,

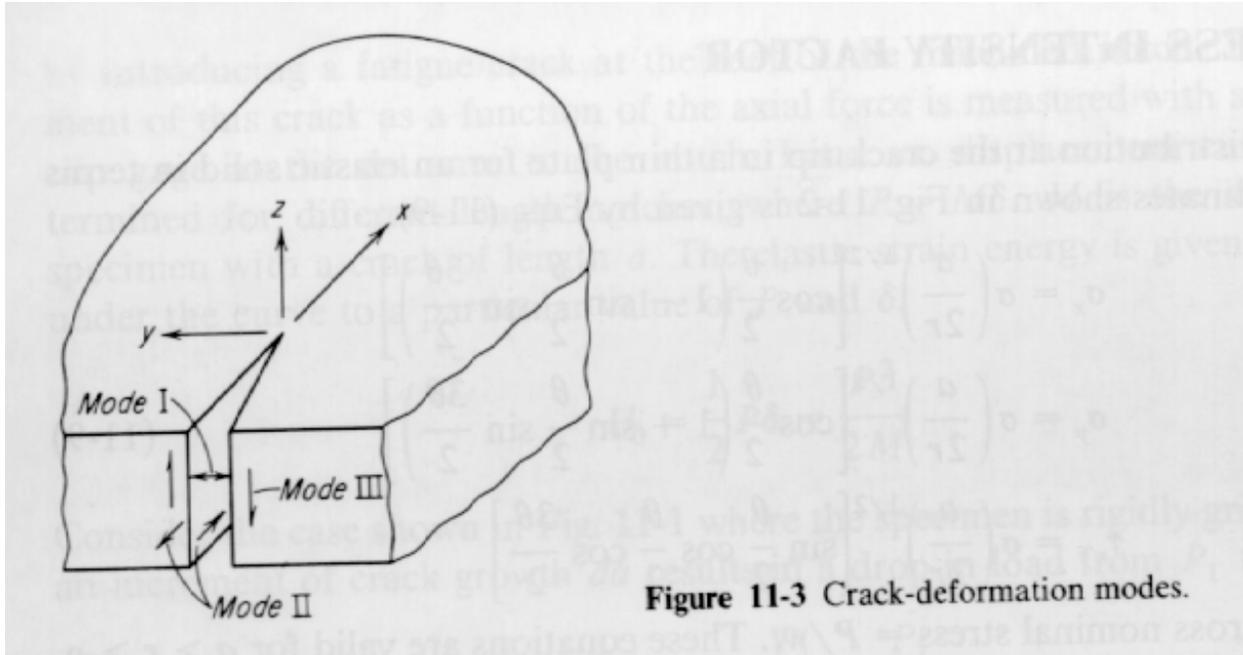
$$x = y = \frac{a}{2r}^{1/2} \quad \text{and} \quad xy = 0$$

The stress intensity factor,  $K$ , is the enhancement at the crack tip of the tensile stress applied normal to the crack, for a sharp flaw in an infinite plate  $K = \sigma \sqrt{\pi a}$ . The stress distribution is usually expressed in terms of this stress intensity factor,  $K$ ,

$$\begin{aligned}
 x &= K \frac{1}{2r}^{1/2} \cos\frac{\theta}{2} \left( 1 - \sin\frac{\theta}{2} \sin\frac{3\theta}{2} \right) \\
 y &= K \frac{1}{2r}^{1/2} \cos\frac{\theta}{2} \left( 1 + \sin\frac{\theta}{2} \sin\frac{3\theta}{2} \right) \\
 xy &= K \frac{1}{2r}^{1/2} \sin\frac{\theta}{2} \cos\frac{\theta}{2} \cos\frac{3\theta}{2}
 \end{aligned}$$

Expressions for the stress intensity factor,  $K$ , for various geometries are given in texts dealing with fracture toughness and some expressions are available in Dieter for instance.

The figure below shows various modes of deformation that could be used to propagate a crack. The most common is mode I and the critical stress intensity factor calculated for this mode would be called  $K_{Ic}$ .



From Dieter.

The stress intensity factor,  $K$ , is directly related to the strain energy release rate,  $G$ ,

$$K^2 = GE \text{ for plane stress}$$

$$K^2 = \frac{GE}{(1 - \nu^2)} \text{ for plane strain}$$

### Plastic Zone at a Crack Tip:

The region at the tip of a crack is subject to ductile yielding in both polymers and metals. We can consider the yield stress,  $\sigma_0$ , as defining plastic zone at the crack tip. The details of yielding depend on the type of material and the microstructure but some generalities can be made.

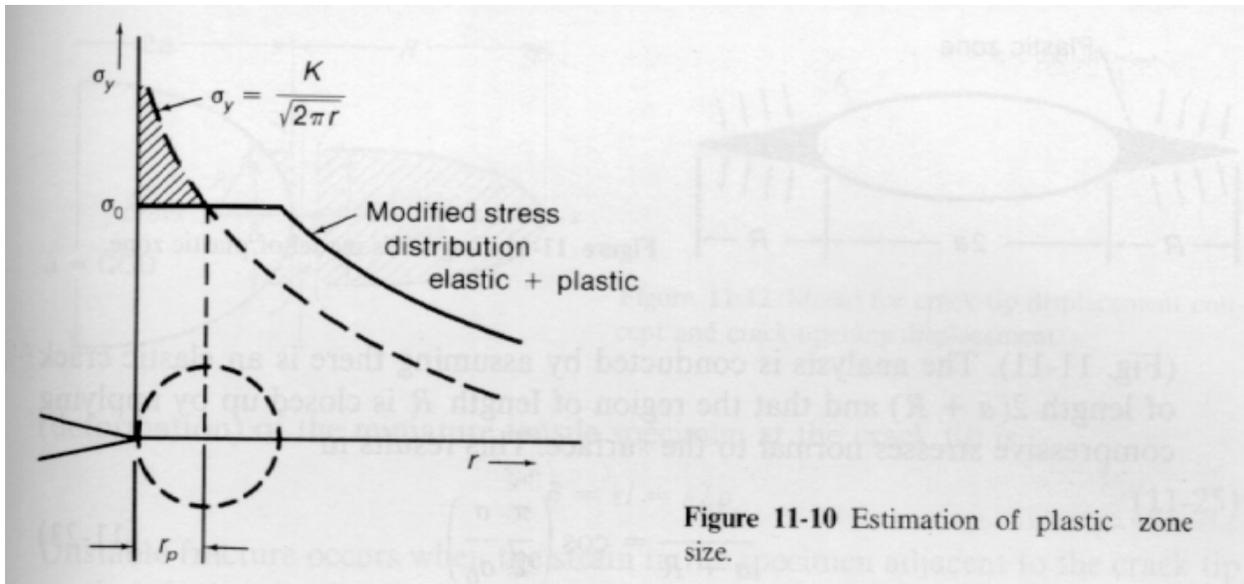


Figure 11-10 Estimation of plastic zone size.

From Dieter.

The figure above from Dieter shows stress as a function of the distance from a crack tip. In the direction  $\theta = 0$  and for the yield stress we have,

$$\sigma_0 = \sigma_y = K \frac{1}{\sqrt{2r_p}} \quad \text{for plane stress}$$

$$\text{so } r_p = \frac{K^2}{2\sigma_0^2} = \frac{a^2}{2}$$

The dashed curve in the figure. The solid curve accounts for elastic and plastic deformation. The radius of the plastic zone is sometimes added to the crack length,  $a$ , to obtain an effective crack length for calculation of the stress intensity factor,  $K$ .

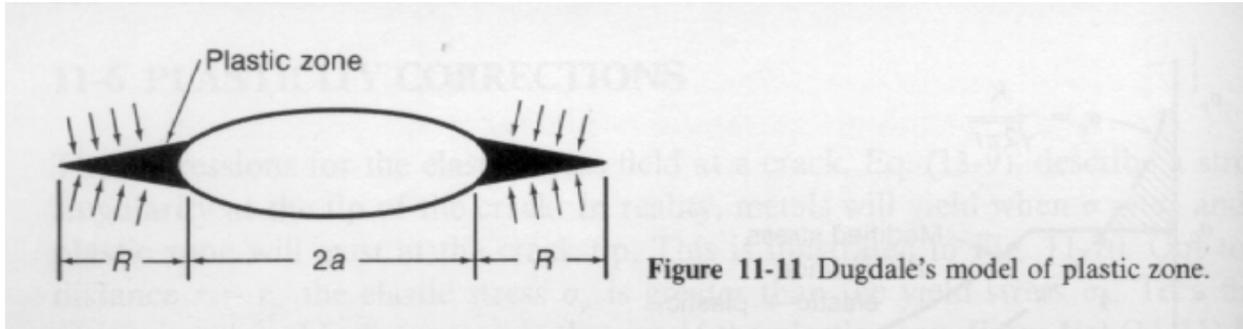
For plane strain the plastic zone is calculated as,

$$r_p = \frac{K^2}{6\sigma_0^2} \quad \text{plane strain}$$

due to differences in the strain field.

The Irwin model, presented above, assumes the plastic zone is spherical or circular and this is clearly not the case. The approach can be refined for a more realistic slit shaped plastic zone, shown below, using the Dugdale model. The Dugdale model includes compressive stresses that act to close the plastic zone, shown in the figure. The plastic zone size in the Dugdale approach is given by,

$$R = \frac{2a}{8} = \frac{K^2}{8}$$



### Extensive Plastic Deformation, $r_p \sim a$ :

For many materials, such as aluminum and polymers, the plastic zone is of comparable size to the original crack. In these cases the perturbative approaches presented above are of less use except as a starting point for consideration. A detailed consideration and a model for the mechanical behavior in the plastic zone is needed and extensive work has been done in this regard. The simplest of these models involves considering the plastic zone as composed of a series of fibers which bridge the plastic zone. Such a morphology actually exists in polymer crazes which are the dominant form of cracks in polymers. The fibers have a length  $l$  and a width  $w$ .  $l$  is determined by the radius of the crack tip,  $l = 2r_p$ , the width is a material feature. For this *crack-tip displacement model*, crack growth involves failure of these fibers in sequence from the center of the crack outwards. For one of these fibers the deformation of the fiber in length,  $\delta$ , is given by the bulk sample strain,  $\epsilon$ ,

$$\delta = l \epsilon$$

Failure of the material occurs at  $\epsilon_f$ , where the fiber displacement reaches a critical value,

$$\delta_c = 2r_{fc}$$

The crack opening displacement model leads to a relationship between the fiber displacement and the strain-energy release rate,  $G$ ,

$$G = \frac{1}{2} \sigma^2 \delta$$

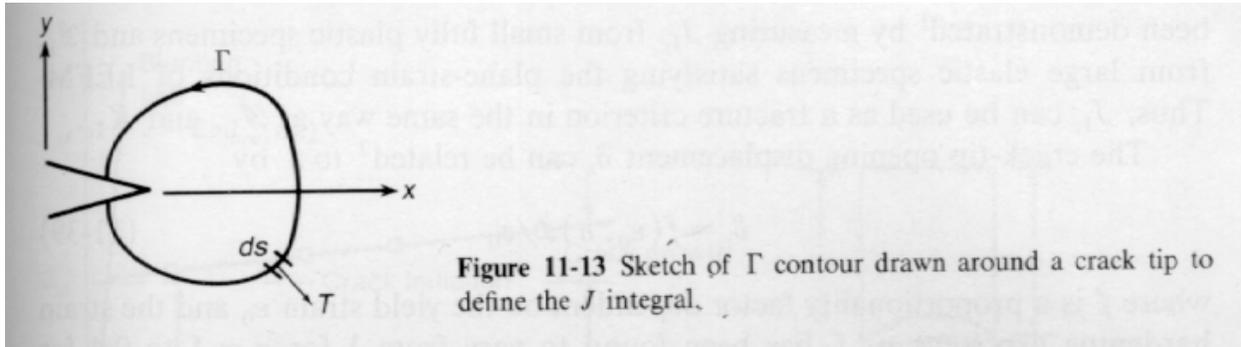
and

$$G_{lc} = \frac{1}{2} \sigma_c^2 \delta_c$$

where  $\delta_c$  is on the order of  $l$  and depends on the extent of the plastic zone.

## J-Integral Method:

The strain-energy release rate for a crack regardless of its degree of plasticity can be determined by considering a line integral of the strain energy per unit volume summed with the integral of the normal stress acting on the contour of the line integral, see figure below from Dieter.



$$J = \int W dy - T \frac{u}{x} ds$$

$W = \int_{ij} d_{ij}$  strain energy per unit volume

$\Gamma$  is the integral path around the crack

$T$  is the normal stress on the integral path

$u$  is the bulk sample displacement

$ds$  is a part of the path

$T \frac{u}{x} ds$  is the rate of work in the area enclosed by

If the  $J$  integral reaches a critical value failure occurs. For example values of  $J_{ic}$  are commonly tabulated for materials. The  $J$  integral is path independent so a path along the sample boundary can be used making analytic measurement of the  $J$  integral a common method in failure analysis. Dieter gives detailed descriptions of ASTM methods for determination of the  $J$  integral.

Consider two samples that are strained to slightly different extents with slightly different crack lengths,  $a$ . The  $J$  integral for this material is equal to the potential energy difference between the two specimens divided by the difference in crack length,

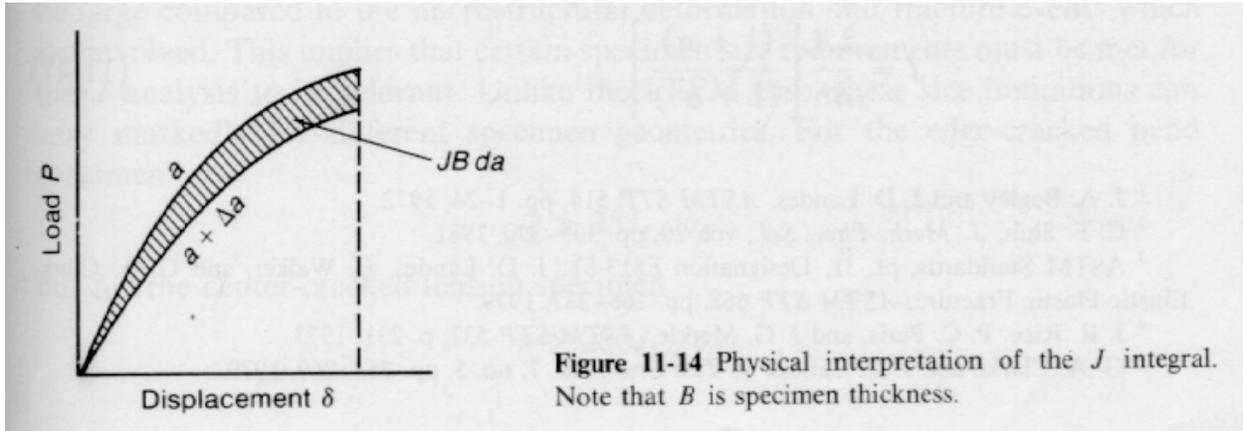
$$J = \frac{U_0}{a} = G = \frac{K^2}{E'}$$

where  $E'$  is defined depending on the sample geometry,

$E' = E$  for plane stress (thick tensile sample)

$E' = \frac{E}{(1 - \nu^2)}$  for plane strain (thin tensile sample)

The figure below illustrates this view of the J integral in a load displacement curve from Dieter,



### Dynamics of Failure in Viscoelastic Materials:

Failure is by nature a dynamic process. That is, failure occurs while a sample is being loaded at some rate. The considerations given above assume that the rate of deformation have little to do with failure since the time constant associated with failure is assumed to be much smaller than the inverse of the strain rate. This is true of many materials. It is not true of most viscoelastic materials that display relaxation times on the order of or larger than the inverse of common strain rates. The rate dependence, for a viscoelastic material following Arrhenius or WLF behavior, translates into a strong temperature dependence for the failure behavior of viscoelastic materials. Details of this behavior will be discussed in the next section.